# Analysis of Structures with Rotating, Flexible Substructures Applied to Rotorcraft Aeroelasticity

Dewey H. Hodges*
*Georgia Institute of Technology, Atlanta, Georgia*
and
A. Stewart Hopkins† and Donald L. Kunz†
*NASA Ames Research Center, Moffett Field, California*

**The General Rotorcraft Aeromechanical Stability Program (GRASP) was developed to perform aeroelastic stability analyses of rotorcraft in steady, axial flight and ground contact conditions. In order to attain the desired level of modeling flexibility and the ability to analyze structures with rotating parts, GRASP has implemented a new method of dynamic structural analysis that combines the finite-element method with multibody dynamics. This method calls for a structure to be decomposed into a hierarchy of substructures, so that discrete relative motion between substructures can be treated exactly. Substructures are modeled using frames of reference, nodes, constraints, and finite elements. The library of finite elements allows a variety of rigid and deformable elements to be included in the analysis. In this paper, the theoretical basis for the implementation of the method is presented, as well as examples of the derivations of element and constraint equations taken from GRASP.**

## Nomenclature

| | |
|---|---|
| $b$ | = basis vector |
| $C$ | = direction cosine array or damping matrix |
| $e$ | = unit vector or eliminated generalized coordinates |
| $F$ | = current frame of reference |
| $G$ | = geometric |
| $g$ | = general constraint relationship function |
| $I$ | = inertial frame of reference |
| $K$ | = stiffness matrix |
| $L$ | = linear operator |
| $M$ | = mass matrix |
| $N$ | = number of |
| $Q$ | = generalized force |
| $q$ | = generalized coordinate |
| $R$ | = position |
| $r$ | = retained generalized coordinates |
| $S$ | = parent frame of reference |
| $\Delta$ | = the identity matrix |
| $\delta$ | = variation of |
| $\delta W$ | = virtual work |
| $\delta\psi$ | = virtual rotation |
| $\epsilon$ | = Levi-Civita symbol |
| $\theta$ | = magnitude of Euler rotation or components of a small rotation |
| $\phi$ | = Euler-Rodrigues parameters |

*Conventions*

| | |
|---|---|
| $\boldsymbol{a}$ | = the Gibbsian vector named $\boldsymbol{a}$ |
| $\hat{\boldsymbol{a}}$ | = the unit vector $\boldsymbol{a}$ |
| $\bar{a}$ | = the steady-state value of $a$ |
| $\breve{a}$ | = the perturbation value of $a$ |
| $\tilde{a}$ | = the associated skew symmetric matrix for the vector $\boldsymbol{a}$ |
| $a'$ | = $a$ after steady-state deformation |
| $a''$ | = $a$ after steady-state deformation and perturbation |
| $a^{bc}$ | = the $a$ associated with $b$ relative to $c$ |
| $a^T$ | = the transpose of the matrix $a$ |
| $a_{b_i}$ | = the $i$th component of the vector $\boldsymbol{a}$ in the $b$ basis |
| $a_i$ | = the $i$th component of the vector $\boldsymbol{a}$ or the $i$th instance of the quantity $a$ |
| $\boldsymbol{a} \times \boldsymbol{b}$ | = vector cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ |

Received Feb. 13, 1987; presented at the Dynamics Specialists' Conference, Monterey, California, April 9-10, 1987; revision received May 8, 1988. This paper is declared a work of the U.S. Government and is not subject to copyright protection in the United States.

*Professor, School of Aerospace Engineering. Associate Fellow AIAA.

†Research Scientist, U. S. Army Aviation Research and Technology Activity, Rotorcraft Dynamics Division, Aeroflightdynamics Directorate. Member AIAA.

## Introduction

OVER the years, rotorcraft aeroelastic stability analyses developed for research purposes have evolved from simple models[1] that were designed to investigate a limited range of physical phenomena into more complex models[2-4] that were intended to accurately represent modern rotorcraft configurations. At the same time, large, sophisticated, rotorcraft simulation programs[5-8] that included many features not present in the research codes were being developed to solve a variety of rotorcraft problems. These programs made significant contributions to the understanding of rotorcraft modeling but, as discussed in Ref. 9, they were not designed with rotorcraft stability as their primary application. Neither do they possess the depth, generality, and modeling flexibility needed to handle the full range of rotorcraft configurations.

Against this background, a requirement emerged for an analysis program that was specifically designed to calculate the aeroelastic stability of rotorcraft and that possessed the modeling capabilities necessary to handle a wide range of configurations. Such a program should be capable of handling large and small problems without artificial size limitations or excessive overhead. It should assemble its models from a library of finite elements in order to assure modeling flexibility. In addition, it must be capable of coupling rotating structures to nonrotating structures. The methodology used to develop such a program is described in Ref. 10. Research in spacecraft dynamics has provided the basis for this methodology, which incorporates body flexibility with the large, discrete motions previously available only in multibody programs.[11] The concept underlying the methodology is that a structure is modeled as a collection of substructures that may undergo discrete motions relative to one another. The substructures themselves are deformable, elastic continua which may be represented by finite elements. Thus, the powerful modeling capabilities available in finite-element methods are combined with the capability of allowing large discrete motions between substructures.

The General Rotorcraft Aeromechanical Stability Program (GRASP) was developed using the methodology from Ref. 10. The initial version of GRASP was designed specifically for the purpose of calculating the aeromechanical stability of rotorcraft in axial flight and ground contact. Configurations or conditions that would result in periodic or otherwise time-dependent motion are not currently supported. GRASP contains libraries for finite elements and constraints from which structural models are built. In addition, it contains a library of solution methods that, in the initial version, includes a steady-state solution and an eigensolution. The steady-state solution is one in which the deformations within each material continuum of the structure remain constant, although there may be discrete motions between the material continua. The eigenproblem solution is one in which the deformations within each material continuum are perturbed from a constant (typically steady-state) condition. Eigenvalues and eigenvectors are obtained from the linearized, perturbation equations. Expansion of these capabilities to other flight regimes can be accomplished by adding solutions to the library.

The primary intent of this paper is to demonstrate how the methodology described in Ref. 10 can be implemented in a rotorcraft aeroelasticity analysis. In so doing, several of the more powerful features included in GRASP are used as examples. A complete presentation of all of the equations for every feature in GRASP is unnecessary for the purposes of this paper, but may be found in the GRASP Theoretical Manual.[12]

## Substructuring

The methodology presented in Ref. 10 is based on the decomposition of a structure into substructures. If discrete motion, such as the rotation of a helicopter rotor, is to be allowed in the structure, then that motion must occur only between substructures (e.g., the rotor and fuselage of a helicopter must be in separate substructures). The concept of substructuring can be extended by creating a hierarchical collection of substructures. That is, the structure is decomposed into substructures, which in turn are decomposed into (sub-)substructures (from here on, sub-substructures will simply be referred to as substructures) and so on. The process is continued until all of the lowest-level substructures are simple finite elements. Decomposing a structure into a hierarchy of substructures provides great flexibility in the way that a structure can be modeled, and provides a powerful organizational tool for structural modeling. In addition to isolating discrete motions within a structure, hierarchical substructuring can be used to separate physical components, to provide a logical separation of substructures by function, to isolate components that will be involved in parametric studies, and to allow natural coordinatization of components.

The hierarchical organization of substructures has been incorporated in GRASP and is implemented as a tree information structure.[13] The root of the tree corresponds to the complete structure with all of the dependent degrees of freedom removed. The root has only one child, which also represents the complete structure and may include some dependent degrees of freedom that have not been constrained. From this point, the tree branches out into substructures that then branch out into other substructures until, at the most detailed level (often called the leaves of the tree), each substructure may be modeled by a single finite element. In GRASP, the relationships among the substructures in the tree are established by defining the substructures in preorder[13] and specifying the identity of the parent substructure within the definition of the substructure. It is important to emphasize that each substructure is a subset of its parent, not a branch of it. Thus, the topology of the structure is not restricted to tree configurations.

In order to avoid any confusion between the subsets of the physical structure (substructures) and the collection of data structures that are used to represent the substructures, GRASP uses the term subsystem to refer to an abstract representation of a substructure. In general, a subsystem is composed of a frame of reference, a set of nodes, a set of generalized coordinates, a set of constraints, and a set of subsystems (some of which are finite elements). The frame of reference is used to keep track of the discrete motion of the substructure relative to its parent, while the nodes represent discrete physical points in the substructure. Those degrees of freedom not associated with a frame or a node are contained in the set of generalized coordinates. The constraints extract the independent generalized coordinates associated with substructure from the collection of all substructure generalized coordinates.

In addition to the subsystem that contains the data describing the characteristics of each substructure, every substructure also has associated with it a state vector, a residual vector, and a set of mass, damping, and stiffness matrices. The components of these arrays correspond to the generalized coordinates of the subsystem. This includes the frame and node degrees of freedom and the generalized coordinates not associated with any frame or node. When the subsystem constraints are applied, only the independent generalized coordinates remain. The independent generalized coordinates for the subsystem then become part of the complete set of generalized coordinates for its parent subsystem. This process continues until, at the root, the components of the arrays correspond to every independent generalized coordinate in the root subsystem.

Hierarchical substructuring, applied in its most general form, would state that a parent substructure owns everything that belongs to each of its children. In practice, however, applying this concept with its full generality would be unwieldy, and would entail substantial overhead. In GRASP, hierarchical substructuring has been implemented such that only the independent generalized coordinates from a child subsystem are available to the parent. For nodes, this means that while the nodal degrees of freedom from the child are passed to parent, the node itself is not. This poses a difficulty when, for instance, a node in a child subsystem is constrained relative to a node in its parent. For a variety of programming reasons and because constraints are more naturally written in a single frame of reference, it is desirable to have the constraint defined in a single subsystem. The problem is solved by creating a node in the child, associated with the same physical region of the material continuum as the node in the parent. Since there are now two identical nodes and two sets of generalized coordinates associated with the same set of physical degrees of freedom, a constraint (called a node demotion constraint) is now required to eliminate the redundant degrees of freedom. In GRASP, the entire process of creating an additional node and its associated node demotion constraint has been designed to be entirely invisible from outside the program.

## Reference Frames

Since equations of motion based on Newtonian mechanics are only valid when referred to an inertial frame of reference, it is essential the there is at least one inertial frame of reference in every model. GRASP satisfies this requirement by making the frame of reference for the root subsystem an inertial frame. Then, if the complete structure is moving at a constant velocity, acceleration, or angular velocity relative to inertial space, the motion can be specified as the motion of the reference frame associated with the child of the root subsystem relative to the root subsystem frame.

In addition to the frames of reference associated with the root subsystem and its child, there is a reference frame associated with every subsystem in the model hierarchy. These subsystem frames of reference are used to represent the discrete motions of the substructures relative to one another. Starting with the child of the root subsystem, the inertial position and orientation of the frame of reference associated with its parent, and the position and orientation of the child frame relative to its parent are known. Given this information, the inertial position and orientation of the child frame can be determined by using the appropriate chain rules. In general,

$$R^{FI} = R^{FS} + R^{SI} \tag{1a}$$

$$C^{FI} = C^{FS}C^{SI} \tag{1b}$$

Therefore, as the model hierarchy is traversed (starting at the root), the position and orientation of every subsystem frame relative to inertial space can be determined.

In a similar manner, the inertial motion of any frame can be calculated since the inertial motion of the parent frame is known, and the position and orientation of the child frame relative to its parent. The chain rules for the calculation of the inertial motion are

$$\Omega^{FI} = \Omega^{FS} + \Omega^{SI} \tag{2a}$$

$$V^{FI} = \Omega^{SI} \times R^{FS} + V^{SI} \tag{2b}$$

$$A^{FI} = \Omega^{SI} \times (\Omega^{SI} \times R^{FS}) + A^{SI} \tag{2c}$$

Each frame of reference, including the one for the root subsystem, introduces six generalized coordinates associated with the rigid-body motions of the frame. For the steady-state problem, the translational generalized coordinates are defined in GRASP to be the measure numbers of the steady-state position of the frame relative to the reference position of the frame, expressed in the basis associated with the steady-state position of the frame $R_{F_i'}^{F'F}$. The rotational generalized coordinates are defined in GRASP to be the Euler-Rodrigues parameters (also called Rodrigues parameters or components of the Rodrigues vector) for the steady-state orientation of the frame relative to the undeformed orientation of the frame $\phi_i^{F'F}$. Euler-Rodrigues parameters were chosen for the rotational generalized coordinates because of the simplicity of converting them to and from direction cosines, and because they are free of singularities for rotations up to 180 deg.

The definition of Euler-Rodrigues parameters used in GRASP differs from the usual definition[14] by a factor of 2, viz.,

$$\phi_i^{BA} = 2\hat{e}_{Ai} \tan\frac{\theta}{2} \tag{3}$$

where a rotation of magnitude $\theta$ occurs about a unit vector $\hat{e}$ (an Euler rotation) that carries the frame $A$ into coincidence with frame $B$. The direction cosine matrix associated with an arbitrary set of those Euler-Rodrigues parameters arranged as a column matrix denoted by $\phi$ is

$$C^{BA} = \left[\left(1 - \frac{\phi^T\phi}{4}\right)\Delta + \frac{\phi\phi^T}{2} - \tilde{\phi}\right] \bigg/ \left(1 + \frac{\phi^T\phi}{4}\right) \tag{4}$$

where the tilde is defined as

$$\tilde{A}_{ij} = -\epsilon_{ijk}A_k \tag{5}$$

and the Levi-Civita symbol $\epsilon_{ijk}$ is 1 if $i,j,k$ is an even permutation of 1,2,3; $-1$ if $i,j,k$ is an odd permutation of 1,2,3,; and 0 otherwise. For an arbitrary change in orientation represented by the direction cosines $C^{BA}$, the virtual rotation is defined as

$$\begin{aligned} \delta C^{BA} &= -\widetilde{\delta\psi}_B^{BA}\, C^{BA} \\ &= -C^{BA}\,\widetilde{\delta\psi}_A^{BA} \end{aligned} \tag{6}$$

The generalized coordinates for the eigenproblem associated with a dynamic system represent infinitesimal perturbations about a set of steady-state values. The dynamic, translational generalized coordinates used in GRASP are $\breve{R}_{F_i''}^{F''F'}$. Since the perturbations are infinitesimal, the rotational generalized coordinates can be associated with the measure numbers of a rotation vector $\breve{\theta}_{F_i''}^{F''F'}$. The direction cosine matrix associated with these generalized coordinates is then

$$C^{F''F'} = \Delta - \tilde{\breve{\theta}}_{F''}^{F''F'} \tag{7}$$

The redundant frame generalized coordinates that appear when a connection is made between two frames are eliminated by frame constraints. In addition, frame constraints establish the position, orientation, and motion of the child frame relative to its parent. Therefore, they also participate in the calculations of frame inertial motion.

To obtain the contributions of frame generalized coordinates to the virtual work, virtual displacements of the frame generalized coordinates are required. The virtual displacements are just variations of the steady-state problem translational and the eigenproblem translational generalized coordinates. The virtual work associated with the steady-state problem rotational and the dynamics problem rotational generalized coordinates must be treated somewhat differently.[10] The virtual work is the dot product of the moment and a virtual rotation vector, the measure numbers of which are $\delta\psi_{F_i'}^{F'F}$ for $\delta\psi_{F_i''}^{F''F'}$ for the steady-state problem and eigenproblem, respectively.

## Nodes

The derivation of the equations of motion for a structure rely heavily on the finite-element method. Just as finite elements are used to discretize the solution field for a complete structure, nodes are used to discretize the solution field for an element. Instead of attempting to solve for the complete element field, nodes are used to define generalized coordinates that approximate the field in the vicinity of the node. GRASP currently supports two types of nodes: structural nodes that define the displacement fields in structural elements, and air nodes that define the airflow fields associated with helicopter rotors.[15] As an example of the implementation of a node, consider the GRASP structural node.

The structural node is designed to model displacement fields that are characterized by small strains, but large displacements and rotations. The displacements are represented by the three traditional translational generalized coordinates. However, the usual approach of taking small rotations about the basis vectors as generalized coordinates is not appropriate for representing large rotations. Therefore, the three rotational generalized coordinates are defined to be the Euler-Rodrigues parameters that specify the orientation of the node after deformation. This set of generalized coordinates is equivalent to the set of generalized coordinates that specify the motion of an infinitesimal, massless, rigid body that occupies the material point associated with the node.

The generalized coordinates for a structural node are, therefore, analogous to the generalized coordinates for a frame of reference. The translational generalized coordinates used in GRASP for the steady-state problem and eigenproblem are $R_{N_i'}^{N'N}$ and $\breve{R}_{N_i''}^{N''N'}$, respectively. The rotational generalized coordinates for the steady-state problem and the eigenproblem are $\phi_{N_i'}^{N'N}$ and $\breve{\theta}_{N_i''}^{N''N'}$, respectively. The virtual displacements associated with the translational generalized coordinates are simply variations of the translational generalized coordinates; $\delta\psi_{N_i'}^{N'N}$ and $\delta\psi_{N_i''}^{N''N'}$ are the virtual rotations associated with the rotational generalized coordinates for the steady-state problem and eigenproblem, respectively.

## Elements

The finite-element basis of the methodology described in Ref. 10 encourages the developer of an analysis to include a library of finite elements in his program. After a description of how the equations for any general element are derived, the derivation of the equations for a specific element, the GRASP aeroelastic beam element, will be discussed.

**Element Equations**

The first step in the derivation of the equations for a finite element involves selecting a set of generalized coordinates that discretize the internal displacements of the element. Since the element displacements are also represented by nodal coordinates, a transformation between the nodal coordinates and element generalized coordinates must be defined. In the case of an element like a rigid-body mass element, these two steps are trivial, since the nodal coordinates are usually chosen to be identical to the element generalized coordinates.

Assuming that the equations are to be derived using the principle of virtual work, expressions for the virtual work of the internal, body, and surface forces are derived. The internal forces arise from deformations of the element itself, whereas the body and surface forces are imposed on the element from outside sources. A rigid-body mass element, for example, is normally subject only to inertial and gravitational body forces.

Once all of the individual contributions to the virtual work have been derived, they are assembled into one complete expression for the virtual work. The equations of motion are then extracted from that statement of virtual work.

**Aeroelastic Beam Element**

As an example of the derivation of the equations for an element, consider the aeroelastic beam element that has been implemented in GRASP. It is the most general element currently included in the GRASP element library, and is used to model slender beams undergoing small strains and large rotations. In the derivation, shear deformation and in-plane distortion of the cross section are ignored. Although warping rigidity is also ignored, all other effects of warping have been retained. The nonlinear beam kinematics for this type of element have been formulated and applied to the dynamic analysis of a pretwisted, rotating, beam element in Ref. 16.

The generalized coordinates of the aeroelastic beam element (Fig. 1) are carried by a frame of reference $F$, two structural nodes $R$ and $T$, and an air node $A$. The axes of both the frame and the root structural node originate at the shear center of the root-end cross section, and are oriented along the cross-section principal axes. Similarly, the tip structural-node axes originate at the shear center of the tip-end cross section, and are oriented along the cross-section principal axes. The air node axes originate on the flowfield axis of symmetry, and are oriented such that the 1-axis lies along the axis of symmetry.

*Spatial Discretization*

The interior displacements of the beam are represented by four functions of the axial coordinate $x_3$: $u_i$ and $\theta_3$. Bending is

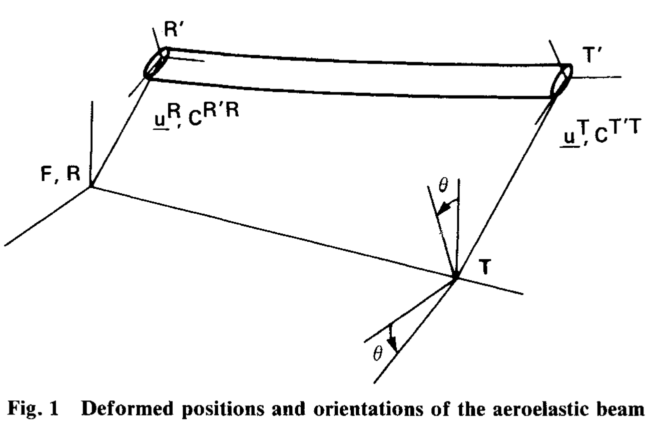


**Fig. 1 Deformed positions and orientations of the aeroelastic beam element relative to its initial state.**

described by $u_1$ and $u_2$, axial displacement by $u_3$, and torsion by $\theta_3$. These functions are discretized in terms of standard cubic and linear polynomials so that the generalized coordinates at the root and tip of the beam can be related to the nodal displacements and rotations. In addition, however, there are also generalized coordinates, termed internal degrees of freedom, associated with higher-order polynomials.

The variables $u_i$ and $\theta_3$ are expanded in a set of polynomials based on Ref. 17. The "C0" functions ($u_3$ and $\theta_3$) are expanded in terms of $\psi_i(x)$ where $x = x_3/\ell$. All functions after the first two standard linear functions are orthonormalized. The "C1" functions $u_\alpha$ are expanded in terms of $\beta_i(x)$. Similarly, all functions after the first four standard cubic functions are orthonormalized.

*Transformation of Coordinates*

Since structural nodes at the root and tip of the beam element have a different set of generalized coordinates than those of the element itself, it is necessary to calculate the beam generalized coordinates in terms of the nodal displacements and rotational variables at both the root and tip of the beam. In general, the beam generalized coordinates are functions of the nodal displacements and the orientation of the tip node relative to the root node.

In addition to the transformation of the nodal coordinates to beam generalized coordinates, the generalized forces associated with the beam generalized coordinates must be transformed into the forces and moments at the root and tip nodes. For the steady-state problem, the transformation is straightforward, but tedious. However, for the dynamics equations of motion, the transformation contributes additional geometric stiffness terms at both the root and tip of the beam. These terms arise because the transformation equations, which must also be perturbed, depend on the nodal coordinates.

*Elasticity*

The elastic equations for the beam[16] are based on the variation of the strain energy

$$\delta U = \int_0^\ell \iint_A (G\epsilon_{3\alpha}\delta\epsilon_{3\alpha} + E\epsilon_{33}\delta\epsilon_{33})\, d\xi_1\, d\xi_2\, dx_3 \tag{8}$$

where

$$\epsilon_{31} = (\lambda_1 - \xi_2)(\kappa_3 - \theta') \tag{9a}$$

$$\epsilon_{32} = (\lambda_2 - \xi_1)(\kappa_3 - \theta') + \bar\epsilon_{33} + \xi_2\kappa_1 - \xi_1\kappa_2 \tag{9b}$$

$$\epsilon_{33} = \tfrac{1}{2}(\xi_1^2 + \xi_2^2)(\kappa_3 - \theta')^2 + (\xi_2\lambda_1 - \xi_1\lambda_2)(\kappa_3 - \theta')\theta' \tag{9c}$$

and where $\theta(x_3)$ is the pretwist angle, with $\theta(0) = 0$, and $(\ )' = d(\ )/dx_3$. The components of the curvature vector $\kappa_i$ depend nonlinearly on $u_\alpha'$ and $\theta_3$, and the extension of the elastic axis $\bar\epsilon_{33}$ depends nonlinearly on $u_i'$. After integrating over the cross-sectional area, we obtain the variation of the strain energy as

$$\delta U = \int_0^\ell (F_3\delta\bar\epsilon_{33} + M_1\delta\kappa_1 + M_2\delta\kappa_2 + M_3\delta\kappa_3)\, dx_3 \tag{10}$$

where $F_3$, $M_1$, $M_2$, and $M_3$ are the stress resultants, which can be expressed in terms of $\bar\epsilon_{33}$ and $\kappa_i$.

*Inertial/Gravitational*

The generalized forces due to the motion of the aeroelastic beam relative to an inertial frame are also derived following Ref. 16. This portion of the derivation, which ignores warp-

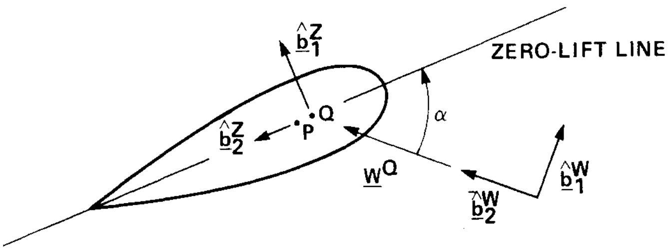


Fig. 2 Coordinate systems for the aeroelastic beam aerodynamics.

ing, is based on the work done by inertial and gravitational forces acting through a virtual displacement. The virtual work is then

$$\delta W = \int_0^\ell (\delta u^P \cdot F^P + \delta\psi^P \cdot M^P)\, dx_3 + \delta u^F \cdot \int_0^\ell F^P\, dx_3 + \delta\psi^F \int_0^\ell (M^P + R^{PF} \times F^P)\, dx_3 \tag{11}$$

where $F^P$ and $M^P$ are, respectively, the inertial and gravitational forces and moments associated with a generic point $P$ along the beam axis, $R^{PF}$ is the position of that point relative to the element frame, $\delta u^P$ and $\delta\psi^P$ are the virtual displacement and virtual rotation at that point, and $\delta u^F$ and $\delta\psi^F$ are the virtual displacement and virtual rotation of the frame.

### *Aerodynamics*

The aerodynamic forces acting on the aeroelastic beam element are determined from a quasisteady adaptation of Greenberg's thin-airfoil theory.[18] Two new sets of coordinate axes must be introduced for the purpose of defining the directions in which the aerodynamic forces and pitching moment act. In Fig. 2, and $Z$ axes are shown to be a set of dextral axes associated with the zero-lift line for the airfoil section. The other set of axes is called the wind axes $W$. For these axes the base vector $\hat{b}_{W_3}$ is identical to $\hat{b}_{Z_3}$. The base vector $\hat{b}_{W_2}$ is along the relative wind vector (in the direction of drag), and $\hat{b}_{W_1}$ is in the direction of lift. As with other axes, these axes convect with the local beam cross section during deformation.

The aerodynamic forces and pitching moment act at the aerodynamic center $Q$ (which is assumed to be the quarter-chord). The offset position of $Q$ relative to the origin of the local principal axes $P$ is $R^{QP}_{Z_2}\hat{b}_{Z_2}$. The dynamic wind velocity vector at the aerodynamic center denoted by $W^{Q''}$ is calculated by subtracting the inertial structural velocity at $Q''$ from the inertial air velocity at $Q''$.

The angle of attack is determined from

$$\tan\alpha = \frac{W^{Q''}_{Z_1''}}{W^{Q''}_{Z_2''}} \tag{12}$$

The local airflow velocity gradient is expressed as

$$G^{Q''}_{Z_{12}''} = \frac{\partial W^{Q''}_{Z_1''}}{\partial R^{QP}_{Z_2}} \tag{13}$$

The relative wind velocity magnitude is given by

$$W = \sqrt{(W^{Q''}_{Z_1''})^2 + (W^{Q''}_{Z_2''})^2} \tag{14}$$

Both the relative wind velocity and the angle of attack are time-dependent quantities that depend on the kinematical variables for the aeroelastic beam element, including frame motion and induced inflow velocity degrees of freedom. The static and dynamic perturbations of these quantities are determined and used in GRASP in their exact form to calculate the generalized forces.

The applied force on the beam is assumed to be

$$F = \mathcal{L}_c \hat{b}_{W_1''} + \mathfrak{D}\hat{b}_{W_2''} + \mathcal{L}_{nc}\hat{b}_{Z_1''} \tag{15}$$

and

$$M = \mathfrak{M}\hat{b}_{Z_3''} \tag{16}$$

The governing equations for the components of the aerodynamic forces and moments are adapted from thin-airfoil theory with arbitrary lift, drag, and pitching moment coefficients $c_l$, $c_d$, and $c_m$.

$$\mathcal{L}_c = {}^1\!/_2\rho_a W^2 c c_l + \frac{\pi}{2}\rho_a c^2 W G^{Q''}_{Z_{12}''} \tag{17a}$$

$$\mathfrak{D} = {}^1\!/_2\rho_a W^2 c c_d \tag{17b}$$

$$\mathfrak{M} = {}^1\!/_2\rho_a W^2 c^2 c_m - \frac{\pi}{16}\rho_a c^3\left(W G^{Q''}_{Z_{12}''} + \dot{W}^{Q''}_{Z_1''} + \frac{3c}{8}\dot{G}^{Q''}_{Z_{12}''}\right) \tag{17c}$$

$$\mathcal{L}_{nc} = \frac{\pi}{4}\rho_a c^2 (\dot{W}^{Q''}_{Z_1''} + \frac{c}{4}\dot{G}^{Q''}_{Z_{12}''}) \tag{17d}$$

With the assumption that the air exerts a force on the structure that is equal and opposite to that of the structure on the air, the virtual work for a dynamic perturbation can be formed as

$$\delta W = \int_0^\ell (-\delta S^{Q''}_{Z_\beta''} F_{Z_\beta''} + \delta\Upsilon^{Q''}_{Z_3''}\mathfrak{M})\, dx_3 \tag{18}$$

where the $\delta S^{Q''}_{Z_3''}$ and $\delta\Upsilon^{Q''}_{Z_3''}$ are the virtual displacements and rotation of an element of air relative to the structure, respectively. These virtual displacements and rotations include all of the kinematical variables for the beam element and degrees of freedom representing frame motion and dynamic inflow velocity and velocity gradients. The virtual work per unit of beam element length done by the aerodynamic forces and pitching moment can then be put into the form of steady-state generalized forces and linear coefficient matrices for generalized accelerations, velocities, and displacements.[15]

## Constraints

Constraints provide the connections among the components (frames, nodes, and elements) of a model. Just as the element library contains a variety of elements that model different pieces of a structure, the constraint library contains a variety of constraints, each with a specific purpose. Some constraints connect frames to one another; others connect nodes to one another. There are also constraints that are used to connect elements to the rest of the structure. In spite of the diversity in the functions of the constraints, all of the constraints have certain features in common. Therefore, it is possible to describe a general method for deriving constraint equations. After discussing constraint equations in general, the derivation of the screw constraint will be presented as an illustration of the method. The screw constraint was selected as the example constraint because it highlights the important role of geometric stiffness in dynamical analyses containing nonlinear constraints.

### Derivation of a General Constraint

In general, a constraint creates dependencies among generalized coordinates. The dependencies are then used to eliminate redundant (dependent) generalized coordinates. In this way, only the independent generalized coordinates in a model are involved in the solution process, thus reducing the size of the solution set. The constraint relationship among generalized coordinates can be written in the general form

$$q_{e_i} = g_i(q_{r_1},\ldots,q_{r_{N^r}}), \qquad (i = 1,\ldots,N^e) \tag{19}$$

Therefore, the generalized coordinates related to the constraint can be partitioned into two sets. And, the set to be eliminated can be obtained directly from the constraint functions, which only depend on the set to be retained. All of the generalized coordinates in a subsystem can be obtained using the constraint relationships to retrieve the eliminated generalized coordinates from the retained generalized coordinates.

The virtual work for the generalized coordinates associated with the constraint is

$$\delta W = \sum_{i=1}^{N^e} \delta q_{e_i} Q_{e_i} + \sum_{i=1}^{N^r} \delta q_{r_i} Q_{r_i} \tag{20}$$

While seeking an equilibrium solution, the sum of the generalized forces associated with a generalized coordinate may differ from zero for two reasons. First, equilibrium may not be satisfied, in which case the residual force is a measure of the error in the approximate solution. Second, even if the system is in equilibrium, an individual substructure may not be in equilibrium. For example, the node that serves as the root of a beam element may not be in equilibrium at the element level, but will be in equilibrium at a higher level when the contributions from all of the other elements connected to that node have been included.

Taking the variation of Eq. (19) yields

$$\delta q_{e_i} = \sum_{j=1}^{N^r} \frac{\partial g_i}{\partial q_{r_j}} \delta q_{r_j}, \qquad (i = 1,\ldots,N^e) \tag{21}$$

Substituting Eq. (21) into Eq. (20) yields

$$\delta W = \sum_{j=1}^{N^r} \delta q_{r_j} \left( Q_{r_j} + \sum_{i=1}^{N^e} \frac{\partial g_i}{\partial q_{r_j}} Q_{e_i} \right) \tag{22}$$

The relationship expressed in Eq. (22) is used to incorporate the contributions of the generalized forces associated with the eliminated generalized coordinates into the retained generalized forces. During the calculation of residual forces, the residuals (from the child subsystem) associated with the eliminated generalized coordinates are transformed and added to the corresponding residuals in the parent subsystem.

The derivation of the constraints for eigenproblem equations of motion is somewhat more involved. In this case, the generalized coordinates are assumed to be the sum of an equilibrium value and an infinitesimal perturbation (i.e., $q = \bar{q} + \check{q}$). Equations (19), (21), and the generalized forces $Q$ can all be expanded in a Taylor series about the steady-state value. Noting that Eq. (19) is valid when $q = \bar{q}$, expansion of Eq. (19) yields

$$\check{q}_{e_i} = \sum_{j=1}^{N^r} \overline{\frac{\partial g_i}{\partial q_{r_j}}} \check{q}_{r_j} + \ldots \qquad (i = 1,\ldots,N^e) \tag{23}$$

Expansion of Eq. (21) yields

$$\delta q_{e_i} = \sum_{j=1}^{N^r} \delta q_{rj} \left( \overline{\frac{\partial g_i}{\partial q_{r_j}}} + \sum_{k=1}^{N^r} \overline{\frac{\partial^2 g_i}{\partial q_{r_j} \partial q_{r_k}}} \check{q}_{rk} + \ldots \right)$$
$$(i = 1,\ldots,N^e) \tag{24}$$

Expansion of the generalized force $Q$, including both eliminated and retained terms, yields

$$Q_i = \bar{Q}_i + \sum_{j=1}^{N} L_{ij} \check{q}_j, \qquad (i = 1,\ldots,N) \tag{25}$$

where the linear operator $\bar{L}$ contains the terms normally associated with the mass, damping, and stiffness matrices $-M(\mathrm{d}^2/\mathrm{d}t^2) - C(\mathrm{d}/\mathrm{d}t) - K$. Note that the minus signs are present in the definition of $\bar{L}$ because the generalized force is generally regarded as positive on the right-hand side of the dynamical equation, whereas the linear coefficient matrices are regarded as positive on the left-hand side.

The method used in GRASP to calculate the $M$, $C$, and $K$ matrices associated with a subsystem involves adding the matrices associated with each of its children to the parent matrices. The rows and columns of the child matrices correspond to all of the generalized coordinates of the child. The constraints are used to eliminate dependent generalized coordinates, resulting in matrices whose rows and columns correspond to the retained generalized coordinates of the child. The elements of these matrices are then transformed into the parent subsystem, and added to those coefficients of the parent matrices that correspond to the child's independent generalized coordinates. The transformations between the parent and child subsystems are obtained form a statement of virtual work.

A statement of the virtual work for perturbations about an equilibrium state may be obtained by substituting Eq. (24), and the eliminated and retained subsets of Eq. (25) into the virtual work expression in Eq. (20). After discarding terms of second order and higher, only a constant term and two first-order terms in $\check{q}$ remain. The constant first term is the same as Eq. (22) evaluated in the equilibrium state.

The linear second term is

$$\sum_{j=1}^{N^r} \sum_{k=1}^{N^r} \delta q_{r_j} \left( \sum_{i=1}^{N^e} \overline{\frac{\partial^2 g_i}{\partial q_{r_j} \partial q_{r_k}}} \bar{Q}_{e_i} \right) \check{q}_{r_k} = \sum_{j=1}^{N^r} \sum_{k=1}^{N^r} \delta q_{r_j} (-K^G_{r_j r_k}) \check{q}_{r_k} \tag{26}$$

The matrix $K^G$ in Eq. (26) represents the geometric stiffness associated with the constraint. During assembly of the matrices associated with the parent subsystem, GRASP calculates this geometric stiffness, and adds it to the stiffness matrix for the parent subsystem. This term is often overlooked, but is extremely important. For example, a pendulum modeled as a rigid-body mass and constrained to rotate about an offset axis using a screw constraint derives *all* of its stiffness from the geometric stiffness term that comes from the constraint.

The linear third term is

$$\sum_{j=1}^{N^r} \sum_{k=1}^{N^r} \delta q_{r_j} \left( \bar{L}_{r_j k} + \sum_{i=1}^{N^e} \overline{\frac{\partial g_i}{\partial q_{r_j}}} \bar{L}_{e_i k} \right) \check{q}_k \tag{27}$$

Separating the sum over $k$ into partial sums involving only eliminated or retained generalized coordinates, and substituting Eq. (23) into Eq. (27) for the eliminated perturbation coordinates, yields

$$\sum_{j=1}^{N^r} \sum_{k=1}^{N^r} \delta q_{r_j} \left( \bar{L}_{r_j r_k} + \sum_{i=1}^{N^e} \overline{\frac{\partial g_i}{\partial q_{rk}}} \bar{L}_{e_i r_k} + \sum_{l=1}^{N^e} \bar{L}_{r_j e_l} \overline{\frac{\partial g_l}{\partial q_{r_k}}} + \sum_{i=1}^{N^e} \sum_{l=1}^{N^e} \overline{\frac{\partial g_i}{\partial q_{r_j}} \frac{\partial g_l}{\partial q_{r_k}}} \bar{L}_{e_i e_l} \right) \check{q}_{r_k} \tag{28}$$

The quantity within the parentheses in Eq. (28) can be thought of as the definition of a new set of $M$, $C$, and $K$ matrices that are expressed in terms of the retained and eliminated partitions of the original matrices. In GRASP, these matrices are calculated, and their coefficients are added to the coefficients of the matrices associated with the parent subsystem.

The complete definition of a constraint then follows from the specification of $g$. An understanding of the effect of the constraint in the dynamics, however, cannot be obtained without explicit calculation of the matrix $\partial g / \partial q$ and the geometric stiffness matrix $K^G$.

### Screw Constraint

Consider two nodes connected by a mechanism that permits translation and rotation, as shown in Fig. 3. Specifically, one node is permitted to displace relative to the other along an axis called the screw axis $\hat{e}^{scr}$. The screw axis is fixed in the deflected and rotated coordinate systems of both nodes. Also, one node is permitted to rotate relative to the other about that same axis. The dependent node (whose generalized coordinates are to be eliminated) is called $D$, and the independent node (whose generalized coordinates are to be retained) is called $I$. Since the $I$ generalized coordinates are known, a relation must be derived that expresses the $D$ generalized coordinates in terms of the $I$ generalized coordinates. To simplify the algebra, two intermediate nodes $S$ and $M$ (for "stationary" and "moving") are introduced and placed on the screw axis. $I$ and $S$ are locked together during deformation of the substructures to which they are attached, as are $D$ and $M$. $M$ and $S$ initially coincide positionally, but may differ in orientation. $R_I^{DI}$, $C^{DI}$, $R_D^{DM}$, and $R_I^{SI}$ are assumed to be given.

*Steady-State Formulation*

For the steady-state problem, the relationships among the generalized coordinates and the contributions of the force and moment acting at $D'$ to those at $I'$ must be written. The displacement and orientation relationships that prescribe the position and orientation of the dependent node in terms of the independent node are

$$R^{D'D} = R^{D'M'} + R^{M'S'} + R^{S'I'} + R^{I'I} + R^{IS} + R^{SM} + R^{MD}$$

$$C^{D'D} = C^{D'M'}C^{M'\bar{M}'}C^{\bar{M}'S'}C^{S'I'}C^{I'I}C^{ID} \tag{29}$$

where $\bar{M}'$ is a node whose position and orientation relative to $S'$ is the same as that of $M$ relative to $S$. In order to obtain the component form of the kinematical relation corresponding to the function $g$ above, the first of Eqs. (29) are written in component form. Let $R^{M'S'} = u\hat{e}^{scr'}$ where $u$ is the screw displacement and $R^{MS} = 0$. In the $D$ basis, the position of $D'$ relative to $D$ is

$$R_D^{D'D} = C^{DI}C^{II'}C^{I'S'}C^{S'\bar{M}'}C^{\bar{M}'M'}C^{M'D'}R_{D'}^{D'M'} - R_D^{DM} + C^{DI}C^{II'}C^{I'S'}\, u e_{S'}^{scr} + C^{DI} (C^{II'}R_{I'}^{S'I'} + R_I^{I'I} - R_I^{SI}) \tag{30}$$

The equations may be simplified somewhat since $C^{IS} = C^{I'S'} = C^{D'M'} = C^{DM} = \Delta$. Also, $C^{M'\bar{M}'}$ may be conveniently expressed as an Euler rotation, where $\theta$ is the screw rotation, and the unit vector about which the rotation occurs is $e_{S'}^{scr'} = e_{\bar{M}'}^{scr'}$. The second of Eqs. (29) is implemented such that each of the direction cosine arrays on the right-hand side is determined first. All of the direction cosine arrays are given in terms of the geometrical parameters for the screw except for $C^{M'\bar{M}'}$, which is known in terms of the screw rotation angle $\theta$, and $C^{I'I}$, which is known in terms of the Euler-Rodrigues parameters for node $I$. Once $C^{D'D}$ is known, the Euler-Rodrigues parameters at $D$ can be calculated.

The generalized forces for this constraint are the components of the actual force and moment at $I'$ and the screw force and moment. In order to calculate these forces, the virtual displacement and virtual rotation components must be determined. The virtual displacement components are the same as the Lagrangian variation of the vector components:

$$\delta R_D^{D'D} = C^{DI}[\delta R_I^{I'I} - (\tilde{R}_I^{D'M'} + u\tilde{e}_I^{scr'} + \tilde{R}_I^{S'I'})\delta\psi_I^{I'I} + e_I^{scr'}\,\delta u + \tilde{e}_I^{scr'}R_I^{D'M'}\,\delta\theta] \tag{31}$$

The virtual rotation components are then

$$\delta\psi_D^{D'D} = C^{DI}(\delta\psi_I^{I'I} + e_I^{scr'}\,\delta\theta) \tag{32}$$

A force and moment at $D'$ will contribute to the virtual work at the screw connection (through $\delta u$ and $\delta\theta$) and at $I'$ (through $R_I^{I'I}$ and $\delta\psi_I^{I'I}$). The virtual work done by $F^{D'}$ and $M^{D'}$ is

$$\delta W = (\delta R_I^{I'I})^T F_I^{D'} + (\delta\psi_I^{I'I})^T(\tilde{R}_I^{D'I'}F_I^{D'} + M_I^{D'}) + \delta u (e_I^{scr'})^T F_I^{D'} + \delta\theta (e_I^{scr'})^T(\tilde{R}_I^{D'M'}F_I^{D'} + M_I^{D'}) \tag{33}$$

Clearly, the force components at $I'$ are simply the components at $D'$ transformed into the $I$ basis. The moment components at $I'$, however, include not only the moment at $D'$ transformed into the $I$ basis, but also the moment of the force at $D'$ about $I'$ in the $I$ basis. The screw force is the component of the force at $D'$ along the screw axis of the deformed structure. Similarly, the screw moment is the component of the moment at $D'$ along the screw axis, augmented by the moment of the force at $D'$ about the screw axis.

The $\partial g/\partial q$ matrix is then

$$\begin{bmatrix} C^{DI} & -C^{DI}\tilde{R}_I^{D'I'} & C^{DI}e_I^{scr'} & -C^{DI}\tilde{R}_I^{D'M'}e_I^{scr'} \\ 0 & C^{DI} & 0 & C^{DI}e_I^{scr'} \end{bmatrix} \tag{34}$$

where the columns of the matrix are associated with the variations of the generalized coordinates $\delta R_I^{I'I}$, $\delta\psi_I^{I'I}$, $\delta u$, and $\delta\theta$, and the rows of the matrix are associated with $\delta R_D^{D'D}$ and $\delta\psi_D^{D'D}$.

*Dynamic Formulation*

For the case of dynamic perturbations, a relationship similar to the one governing the generalized coordinates for the steady-state problem is used to find the matrices $\partial g/\partial q$ and $K^G$. Consider the nodes and the screw axis in their dynamic states, perturbed infinitesimally from their steady-state positions and orientations. The equations for the positions are similar to those for the steady-state case

$$R^{D''D'} = R^{D''M''} + R^{M''\bar{M}''} + R^{\bar{M}''S''} + R^{S''I''} + R^{I''I'}R^{I'D'} \tag{35}$$

Using the virtual rotations for the orientations,

$$\delta\psi^{D''D'} = \delta\psi^{D''M''} + \delta\psi^{M''\bar{M}''} + \delta\psi^{\bar{M}''S''} + \delta\psi^{S''I''} + \delta\psi^{I''S'} + \delta\psi^{I'D'} \tag{36}$$

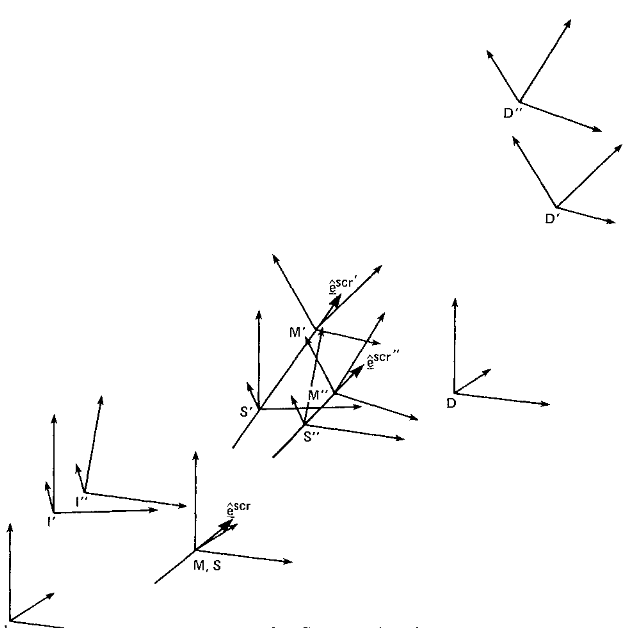


**Fig. 3 Schematic of the screw constraint.**

The first, third, fourth, and sixth terms in Eq. (36) are zero. Proceeding as above and noting that

$$C^{I''I'} = \Delta - C^{I'I}\tilde{\theta}_I^{I''I'}C^{II'} = C^{I'I}(\Delta - \tilde{\theta}_I^{I''I'})C^{II'} \tag{37a}$$

$$\delta C^{I''I'} = -C^{I''I}(\Delta - \tilde{\theta}_I^{I''I'})\widetilde{\delta\psi}_I^{I''I'}C^{II'} \tag{37b}$$

$$\delta C^{M''\bar{M}''} = -\delta\theta\tilde{e}_{M''}^{\text{scr}''}C^{M''\bar{M}''} \tag{37c}$$

the $\partial g/\partial q$ matrix for the dynamics turns out to be the same as for the steady-state problem (34). It should be noted, however, that if the choice of coordinate bases for the nodes had been different (e.g., the basis used for the frames[12]), the matrix $\partial g/\partial q$ probably would not be the same for both the steady-state and dynamic formulations.

The geometric stiffness matrix $K^G$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\tilde{F}_I^{D'}\tilde{R}_I^{D'I'} & \tilde{F}_I^{D'}e_I^{\text{scr}'} & -\tilde{F}_I^{D'}\tilde{R}_I^{D'M'}e_I^{\text{scr}'} \\ 0 & -(e_I^{\text{scr}'})^T\tilde{F}_I^{D'} & 0 & 0 \\ 0 & \begin{matrix} -(e_I^{\text{scr}'})^T\tilde{R}_I^{D'M'}\tilde{F}_I^{D'} \\ -(e_I^{\text{scr}'})^T\tilde{M}_I^{D'} \end{matrix} & 0 & \begin{matrix} -(R_I^{D'M'})^T\tilde{e}_I^{\text{scr}'} \\ \tilde{e}_I^{\text{scr}'}F_I^{D'} \end{matrix} \end{bmatrix} \tag{38}$$

where the columns correspond to the perturbations of the generalized coordinates $\breve{R}_I^{I''I'}$, $\breve{\theta}_i^{I''I'}$, $\breve{u}$, and $\breve{\theta}$, and the rows correspond to the variations of the generalized coordinates $\delta R_I^{I''I'}$, $\delta\psi_I^{I''I'}$, $\delta\breve{u}$, and $\delta\breve{\theta}$.

Note that the geometric stiffness matrix in Eq. (38) is not symmetric. This becomes evident when the $3 \times 3$ rotation-rotation term associated with the independent node is examined:

$$-(\delta\psi_I^{I''I'})^T\tilde{F}_I^{D'}\tilde{R}_I^{D'I'}\theta_I^{I''I'} \tag{39}$$

The geometric stiffness matrix $K^G$ is not symmetric because of the choice of generalized coordinates for the infinitesimal rotations, and the choice of virtual generalized coordinates for the virtual rotations.[19] It is easily shown that, for example, the use of Euler-Rodrigues parameters and their variations leads to a symmetric geometric stiffness matrix. The choices presented herein were implemented in GRASP because the terms in the rows of the matrix equations corresponding to the virtual rotations are physical moments, whereas the generalized forces associated with the variations of Euler-Rodrigues parameters have no easily identifiable physical significance.

## Concluding Remarks

The methodology introduced in Ref. 10 unites the best features of the finite-element method and multibody dynamics. It combines the flexibility in structural representation available with finite elements, with the ability to allow large, discrete motions between portions of the structure. Several aspects of this approach to structural analysis make it a very powerful and useful tool.

The basis of this methodology is the decomposition of the structure into substructures such that discrete motions between substructures can be accommodated. To take full advantage of this capability, the decomposition should be extended to be a hierarchical decomposition of the structure. Each substructure has associated with it a frame of reference that defines the position, orientation, and motion of the substructure as it is moving relative to the rest of the structure.

The equations of motion for a structure are assembled from the equations of motion derived for individual elements. This assembly process is very similar to that used in traditional finite-element programs, except that the assembly has to take place through several levels of the structural hierarchy. Because of the presence of frames of reference in every subsystem, the element equations include terms that reflect the motion of the frame. In addition, most elements will include the nonlinear effects of large rotations, and support deformations that result in large displacements and rotations. The aeroelastic beam element in GRASP provides good example of the implementation of such a finite element.

The generalized coordinates of the elements are based on physical degrees of freedom that are introduced through nodes. Different types of nodes can be used to introduce special-purpose sets of generalized coordinates. Thus, like the element library, a library of nodes can be built to support a wide variety of applications.

A library of constraints provides a variety of means assembling a structure. Some of these constraints may include frame-to-frame constraints, node-to-node constraints, frame-to-node constraints, and constraints that eliminate degrees of freedom. One of the more powerful constraints incorporated in GRASP is the screw constraint. The screw can be used by itself, or to build a variety of gimballed and hinged connections. Its use in this paper as an example of the implementation of a constraint also demonstrates the importance of nonlinear effects in modeling that result in geometric stiffness.

Using the features of this methodology as a means of overcoming many of the limitations of existing rotorcraft stability analyses, GRASP provides both a nonlinear, steady-state solution and a linearized eigensolution for rotorcraft in axial flight and ground contact conditions. Solutions obtained from calculations made with GRASP have thus far been excellent. Comparisons of GRASP results with experimental results in the nonlinear analysis of a cantilever beam are reported in Refs. 9, 16, and 20. Reference 21 reports on the correlation of GRASP results with experimental data for a torsionally soft rotor.

## Acknowledgment

The first author was supported in his efforts on this paper by grant E-16-697 from the Georgia Tech Research Corporation.

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